

Physics 110B
Homework #2

#1 (Griffiths 9.21)

$$R = \left| \frac{\hat{E}_{oR}}{\hat{E}_{oE}} \right|^2 = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2 = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \left(\frac{1 - \tilde{\beta}^*}{1 + \tilde{\beta}^*} \right) \quad (\text{eq 9.147})$$

$$\tilde{\beta} = \frac{M_1 V_1}{M_2 \omega} \tilde{k}_2 \quad (\text{eq 9.146})$$

$$= \frac{M_1 V_1}{M_2 \omega} (k_2 + i\kappa_2) \quad (\text{eq 9.125})$$

$$k = \omega \sqrt{\frac{\epsilon \mu}{Z}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} + 1 \right)^{1/2}, \quad K = \omega \sqrt{\frac{\epsilon \mu}{Z}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right)^{1/2}$$

(eq 9.126)

Since, $\sigma \gg \epsilon \omega$ because silver is a good conductor

$$k_2 \approx K_2 = \omega \sqrt{\frac{\epsilon \mu_2}{Z}} \sqrt{\frac{\sigma}{\epsilon_2 \omega}} = \sqrt{\frac{\sigma \omega \mu_2}{Z}}$$

$$\text{So, } \beta = \frac{M_1 V_1}{M_2 \omega} \sqrt{\frac{\sigma \omega \mu_2}{Z}} (1+i) = M_1 V_1 \sqrt{\frac{\sigma}{2 M_2 \omega}} (1+i)$$

$$A = M_1 V \sqrt{\frac{\sigma}{2 M_2 \omega}} = M_0 C \sqrt{\frac{\sigma}{2 M_0 \omega}} = C \sqrt{\frac{\sigma \mu_0}{2 \omega}} = 3 \times 10^{-8} \sqrt{\frac{6 \times 10^9 \cdot 4 \pi \times 10^{-7}}{2 (4 \times 10^{15})}} = 29$$

Therefore,

$$R = \left(\frac{1 - \gamma - i\gamma}{1 + \gamma + i\gamma} \right) \left(\frac{1 - \gamma + i\gamma}{1 + \gamma - i\gamma} \right) = \frac{(1 - \gamma)^2 + \gamma^2}{(1 + \gamma)^2 + \gamma^2} = \boxed{.93}$$

93% percent of the light is reflected.

#2 (Griffiths 9.22)

a) $V = \omega/k$, we are told $V = \alpha\sqrt{\lambda}$ where α is a constant

$$\text{thus, } \frac{\omega}{k} = \alpha \sqrt{\frac{2\pi}{\lambda}} \Rightarrow \omega = \alpha \sqrt{2\pi k}$$

$$\text{So, } V_{\text{group}} = \frac{d\omega}{dk} = \frac{\alpha}{2} \sqrt{\frac{2\pi}{\lambda}} = \frac{\alpha}{2} \sqrt{\lambda} = V/2$$

$$\text{Or, } \boxed{V_{\text{group}} = V/2}$$

b) $\Psi(x,t) = A \exp i(px - Et)/\hbar = A \exp i(kx - \omega t)$

$$\Rightarrow \hbar k = p, E = \hbar\omega = P^2/2M$$

$$\text{Therefore, } V = \frac{\omega}{k} = \frac{E/\hbar}{P/\hbar} = \frac{P^2/2M}{P} = \frac{P}{2M} = \boxed{\frac{\hbar k}{2M} \text{ wave velocity}}$$

$$\text{Also, } V_{\text{group}} = \frac{d\omega}{dk} = \frac{d}{dk} \left(\frac{\hbar k^2}{2M} \right) = \boxed{\frac{\hbar k}{M} \text{ group velocity}}$$

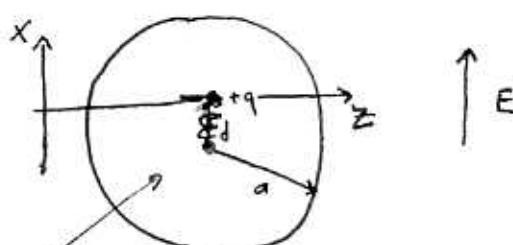
$$= \frac{P}{M} \quad \text{or} \quad P = M V_{\text{group}} = M V_{\text{classical}}$$

Thus,

$$\boxed{V_{\text{group}} = V_{\text{classical}}} , \text{ and } \boxed{V = V_{\text{group}}/2}$$

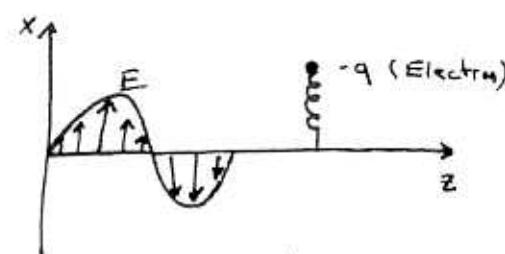
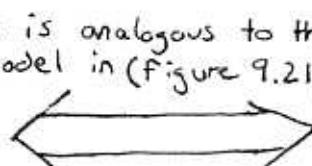
#3 (Griffiths 9.23)

Ferm model in Ex 4.1



$-q$ (electron charge cloud)
distance varies by d .

This is analogous to the
model in (figure 9.21)



Electrons distance
varies by x .

Therefore, from example 4.1

$$E = \frac{1}{4\pi\epsilon_0} \frac{q^2}{a^3} \Rightarrow F = -qE = -\frac{1}{4\pi\epsilon_0} \frac{q^2 x}{a^3} = -k_{\text{spring}} x = -M\omega_0^2 x \quad (\text{eq 9.151})$$

So, $\boxed{\omega_0 = \sqrt{\frac{q^2}{4\pi\epsilon_0 Ma^3}}}$

$$\nu_0 = \omega_0 / 2\pi = \frac{1}{2\pi} \sqrt{\frac{(1.6 \times 10^{-19})^2}{4\pi (8.85 \times 10^{-12})(9.11 \times 10^{-31})(5 \times 10^{-16})^3}} = \boxed{7.16 \times 10^{15} \text{ Hz}}$$

$$\lambda = c / \nu_0 = 3 \times 10^8 / 7.16 \times 10^{15} = 3.7 \times 10^{-7} = 50 \text{ nm}$$

this is in the ultraviolet region
of the spectrum

Now,

$$n = 1 + \frac{Nq^2}{2ME_0} \frac{f_0}{\omega_0^2} \left(1 + \frac{\omega^2}{\omega_0^2} \right) \quad (\text{eq 9.173})$$

$$n = 1 + A \left(1 + \frac{B}{\lambda^2} \right) \quad (\text{eq 9.174})$$

So, $A = \frac{Nq^2}{2ME_0} \frac{f_0}{\omega_0^2}, \quad \begin{cases} N = \# \text{ of molecules per unit vol.} = \frac{6.022 \times 10^{23}}{22.4 \text{ L}} \\ = 6.022 \times 10^{23} / 22.4 \times 10^{-3} \text{ m}^3 = 2.69 \times 10^{25} \end{cases}$

$$= \frac{2.69 \times 10^{25} (1.6 \times 10^{-19})^2}{(9.11 \times 10^{-31})(8.85 \times 10^{-12}) (4.5 \times 10^{-16})^2} \quad \begin{cases} f = \# \text{ of electrons at the natural frequency } \omega_0 \\ = 2 \end{cases}$$

$$= \boxed{4.2 \times 10^{-5}} \quad (\text{which is about } \sqrt{3} \text{ the actual value})$$

$$B = \frac{\omega^2 \lambda^2}{\omega_0^2} = \left(\frac{2\pi c}{\omega_0} \right)^2 = \left(\frac{2\pi \times 3 \times 10^8}{4.5 \times 10^{16}} \right)^2 = \boxed{1.8 \times 10^{-15} \text{ m}^2}$$

(which is about 1/4 the actual value)

#4 (Griffiths 9.24)

In (Figure 9.22) the width of the anomalous dispersion region goes from ω_1 to ω_2 at these two points n is at a maximum and a minimum respectively.

Thus, $n \equiv 1 + \frac{Nq^2}{ZME_0} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$ ← the denominator we label A

Since, ω_1 and ω_2 are maxes and mins we set the derivative of the index of refraction equal to zero

$$\frac{dn}{d\omega} = \frac{Nq^2}{ZME_0} \left\{ \frac{-2\omega}{A} - \frac{\omega_0^2 - \omega^2}{A^2} (2(\omega_0^2 - \omega^2)(-\omega) + \gamma^2 2\omega) \right\} = 0$$

$$2\omega D = (\omega_0^2 - \omega^2)(2(\omega_0^2 - \omega^2) - \gamma^2) 2\omega$$

$$(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2 = 2(\omega_0^2 - \omega^2)^2 - \gamma^2 (\omega_0^2 - \omega^2)$$

$$(\omega_0^2 - \omega^2)^2 = \gamma^2 (\omega^2 + \omega_0^2 - \omega^2) = \gamma^2 \omega_0^2$$

$$\omega_0^2 - \omega^2 = \pm \omega_0 \gamma \quad \gamma \ll \omega_0$$

$$\omega = \sqrt{\omega_0^2 \mp \omega_0 \gamma} = \omega_0 \sqrt{1 + \frac{\gamma}{\omega_0}} \approx \omega_0 (1 + \frac{\gamma}{2\omega_0})$$

Therefore, $\omega_1 = \omega_0 - \gamma/2$, $\omega_2 = \omega_0 + \gamma/2$

And, $\boxed{\Delta\omega = \omega_2 - \omega_1 = \gamma}$

$$\alpha = 2K \approx \frac{Nq^2 \omega^2}{ZME_0 C} \frac{\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \quad (\text{eq 9.171})$$

when

$$(\omega = \omega_0) \quad \alpha_{\max} = \frac{Nq^2}{ZME_0 C \gamma} \quad \text{at } \omega_1 \text{ and } \omega_2, \omega^2 = \omega_0^2 \mp \omega_0 \gamma \text{ respectively}$$

$$\text{So, } \alpha = \frac{Nq^2 \omega^2}{ZME_0 C} \frac{\gamma}{\gamma^2 \omega_0^2 + \gamma^2 \omega^2} = \alpha_{\max} \left(\frac{\omega^2}{\omega^2 + \omega_0^2} \right)$$

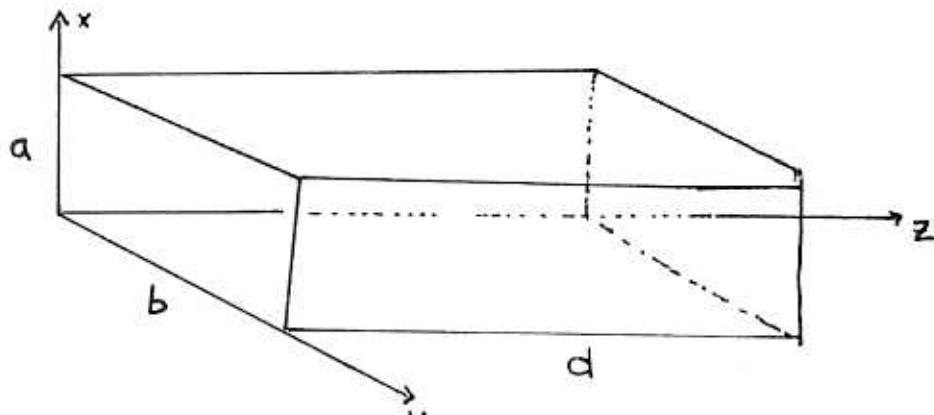
$$\gamma \ll \omega_0$$

$$\text{Also, } \frac{\omega^2}{\omega^2 + \omega_0^2} = \frac{\omega_0^2 + \omega_0 \gamma}{2\omega_0^2 + \omega_0 \gamma} = \frac{1}{2} \frac{(1 + \gamma/\omega_0)}{(1 + \gamma/2\omega_0)} \approx \frac{1}{2} \left(1 + \frac{\gamma}{\omega_0}\right) \left(1 \pm \frac{\gamma}{2\omega_0}\right)$$

$$= \frac{1}{2} \left(1 \pm \frac{\gamma}{2\omega_0} + \frac{\gamma}{\omega_0} - \frac{\gamma^2}{2\omega_0^2}\right) \approx \frac{1}{2} \left(1 + \frac{\gamma}{2\omega_0}\right) \approx \frac{1}{2}$$

Therefore, $\alpha \approx \frac{1}{2} \alpha_{\max}$ at ω_1 and ω_2

#5 (Griffiths 9.38)



Since, we have a resonant cavity we will have standing waves, therefore (eq 9.176) now will read:

$$i) \tilde{E}(x, y, z, t) = \tilde{E}_0(x, y) \exp(i\omega t)$$

$$ii) \tilde{B}(x, y, z, t) = \tilde{B}_0(x, y) \exp(-i\omega t)$$

Thus, we note to get the correct equations for our standing waves we replace

$-ik \Rightarrow \frac{\partial}{\partial z}$ in the equations on page 406-407

We thus have:

$$\left. \begin{array}{l} i) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \right) E_z = 0 \\ ii) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \right) B_z = 0 \end{array} \right\} \text{(eq 9.181)}$$

Looking at transverse magnetic waves $B_z=0$ we solve the above equation by separation of variables:

$$E_z(x, y, z) = X_1(x) Y_1(y) Z_1(z)$$

$$\text{So, } X_1 Y_1 \frac{d^2 Z_1}{dz^2} + Z_1 Y_1 \frac{d^2 X_1}{dx^2} + X_1 Z_1 \frac{d^2 Y_1}{dy^2} + \left(\frac{\omega}{c}\right)^2 = 0$$

$$\text{i) } \frac{1}{X_1} \frac{d^2 X_1}{dx^2} = -k_x^2, \quad \text{ii) } \frac{1}{Y_1} \frac{d^2 Y_1}{dy^2} = -k_y^2, \quad \text{iii) } \frac{1}{Z_1} \frac{d^2 Z_1}{dz^2} = -k_z^2$$

The general solutions to (i), (ii), and (iii) are:

$$X_1(x) = A_1 \sin(k_x x) + B_1 \cos(k_x x)$$

$$Y_1(y) = C_1 \sin(k_y y) + D_1 \cos(k_y y)$$

$$Z_1(z) = E_1 \sin(k_z z) + F_1 \cos(k_z z)$$

By the symmetry of the problem we will get the following similar results for E_x and E_y :

$$E_y(x, y, z) = X_2(x) Y_2(y) Z_2(z)$$

$$X_2(x) = A_2 \sin(k_x x) + B_2 \cos(k_x x)$$

$$Y_2(y) = C_2 \sin(k_y y) + D_2 \cos(k_y y)$$

$$Z_2(z) = E_2 \sin(k_z z) + F_2 \cos(k_z z)$$

$$E_x(x, y, z) = X_3(x) Y_3(y) Z_3(z)$$

$$X_3(x) = A_3 \sin(k_x x) + B_3 \cos(k_x x)$$

$$Y_3(y) = C_3 \sin(k_y y) + D_3 \cos(k_y y)$$

$$Z_3(z) = E_3 \sin(k_z z) + F_3 \cos(k_z z)$$

for
 $E_y(x, y, z)$

for
 $E_x(x, y, z)$

Now, the above results need to meet the boundary condition $E_{||} = 0$

Thus, when $z=0$ and $z=d$; $E_z \neq 0$, $E_y=0$, $E_x=0$

therefore, $F_2 = F_3 = 0$ $E_z = 0$ with $k_z = l\pi/d$

when $y=0$ and $y=b$; $E_y \neq 0$, $E_z=0$, $E_x=0$

So, $D_1 = D_3 = C_2 = 0$ with $k_y = n\pi/b$

Similarly for $x=0$ and $x=a$; $E_x \neq 0$, $E_y=0$, $E_z=0$

$B_1 = B_2 = A_3 = 0$ with $k_x = m\pi/a$

Putting all these results together we have,

$$\bar{E} = B \cos(k_x x) \sin(k_y y) \sin(k_z z) \hat{x} + D \sin(k_x x) \cos(k_y y) \sin(k_z z) \hat{y} + F \sin(k_x x) \sin(k_y y) \cos(k_z z) \hat{z}$$

with
 $k_x = m\pi/a$
 $k_y = n\pi/b$
 $k_z = l\pi/d$

The magnetic field we get from $\bar{B} = -i/\omega \bar{\nabla} \times \bar{E}$

$$\text{Therefore, } B_x = -i/\omega \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)$$

$$B_y = -i/\omega \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)$$

$$B_z = -i/\omega \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

Plugging the electric field from above we get

$$\bar{B} = -i/\omega (F k_y - D k_z) \sin(k_x x) \cos(k_y y) \cos(k_z z) \hat{x} - i/\omega (B k_z - F k_x) \cos(k_x x) \sin(k_y y) \cos(k_z z) \hat{y} - i/\omega (D k_x - B k_y) \cos(k_x x) \cos(k_y y) \sin(k_z z) \hat{z}$$

Also, from the altered eq 9.181 at the beginning of this problem we have:

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = c^2 \left(\left(M\pi/a \right)^2 + \left(n\pi/b \right)^2 + \left(l\pi/d \right)^2 \right)$$

Or,

$$\boxed{\omega = c\pi \sqrt{\left(M/a \right)^2 + \left(n/b \right)^2 + \left(l/d \right)^2}}$$

#6 (Griffiths 9.31)

part (b) only

For a coaxial transmission line: $E(s, \varphi, z, t) = \frac{A \cos(kz - \omega t)}{s} \hat{s}$

$B(s, \varphi, z, t) = \frac{A \cos(kz - \omega t)}{cs} \hat{\varphi}$

$\left. \right\} \text{eq 9.197}$

To determine $\lambda(z, t)$ we use Gauss's law

$$\oint E \cdot d\alpha = \frac{\cos(kz - \omega t)}{s} 2\pi s \cdot z = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \lambda dz$$

$$\Rightarrow \boxed{\lambda(z, t) = 2\pi\epsilon_0 A \cos(kz - \omega t)}$$

To determine $I(z, t)$ we use Ampere's Law

$$\oint B \cdot d\ell = \frac{A}{c} \frac{\cos(kz - \omega t)}{s} 2\pi s = \mu_0 I_{\text{enc}}$$

$$\Rightarrow \boxed{I(z, t) = \frac{2\pi A}{\mu_0 c} \cos(kz - \omega t)}$$

The charge on the outer conductor are the [opposite] of these and current since there is no E or B inside the metal.

#7 From the previous problem we have the following results

$$\bar{E} = \frac{A \cos(kz - \omega t)}{s} \hat{s}$$

$$\bar{B} = \frac{A \cos(kz - \omega t)}{cs} \hat{\phi}$$

$$\lambda = 2\pi\epsilon_0 A \cos(kz - \omega t), \quad I = \frac{2\pi A}{\mu_0 c} \cos(kz - \omega t)$$

$$\begin{aligned} \Delta V &= - \int_b^a \bar{E} \cdot d\bar{l} = - \int_b^a \frac{A}{s} \cos(kz - \omega t) \hat{s} \cdot d\bar{s} \\ &= -A \cos(kz - \omega t) \int_b^a \frac{ds}{s} = A \cos(kz - \omega t) \ln \frac{b}{a} \end{aligned}$$

$$\text{Thus, } Z_o \equiv \frac{\Delta V}{I} = \frac{A \cos(kz - \omega t) \ln \frac{b}{a}}{\frac{2\pi A}{\mu_0 c} \cos(kz - \omega t)} = \frac{1}{2\pi} \frac{1}{\mu_0 c} \ln \frac{b}{a}$$

$$Z_o = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \frac{b}{a}$$

Now,

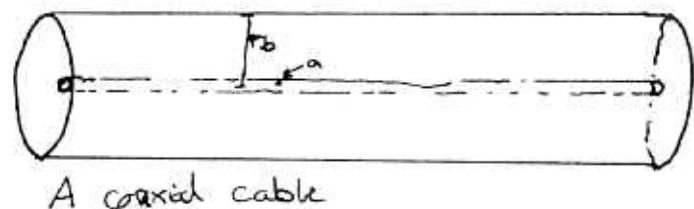
$$W = \frac{1}{2} L I^2 = \frac{1}{2\mu_0} \int B^2 d\tau = \frac{1}{2\mu_0} \int_a^b \frac{A^2 \cos^2(kz - \omega t)}{c^2 s^2} 2\pi s l ds$$

$$= \frac{1}{2\mu_0} \frac{A^2 \cos^2(kz - \omega t)}{c^2} 2\pi l \ln \frac{b}{a}$$

$$L = \frac{1}{2\mu_0} \frac{A^2 \cos^2(kz - \omega t)}{c^2} 2\pi l \ln \frac{b}{a} \cdot 2 \left(\frac{2\pi A}{\mu_0 c} \cos(kz - \omega t) \right)^2$$

$$= \frac{1}{2\mu_0} \frac{1}{2\pi} \frac{\mu_0^2}{c^2} l \ln \frac{b}{a} 2$$

$$= \frac{\mu_0 \ln \frac{b}{a}}{2\pi} l \quad \Rightarrow \quad L' = \frac{\mu_0 \ln \frac{b}{a}}{2\pi}$$

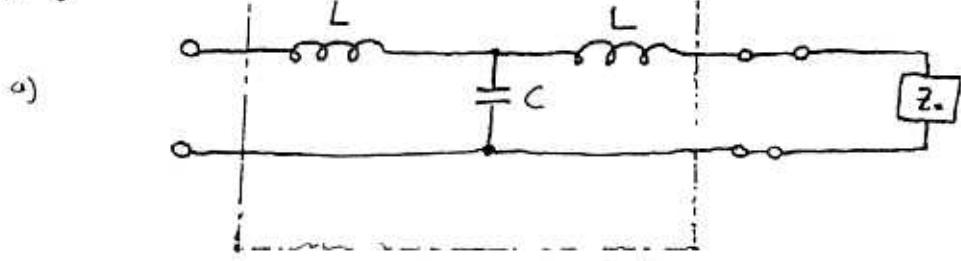


$$C' = \frac{Q}{\Delta V} \frac{1}{L} = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0 A \cos(\kappa z - \omega t)}{A \cos(\kappa z - \omega t) \ln b/a} = \boxed{\frac{2\pi\epsilon_0}{\ln b/a}}$$

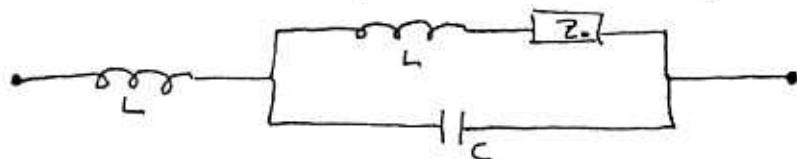
Thus,

$$Z_0 = \sqrt{\frac{L}{C'}} = \sqrt{\frac{\mu_0 \ln b/a}{2\pi}} / \frac{2\pi\epsilon_0}{\ln b/a} = \boxed{\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\ln b/a}{2\pi}}$$

#8



The above circuit is equivalent to the following circuit,



Now, we wish to find the effective impedance of the above arrangement.

$$\underbrace{Z_0 + Z_L}_{\text{in series}} \Rightarrow Z_1 = Z_0 + Z_L$$

$$\text{Next, } \underbrace{\frac{Z_1 Z_C}{Z_1 + Z_C}}_{\text{in parallel}} \Rightarrow Z_2 = \frac{Z_1 Z_C}{Z_1 + Z_C}$$

Finally,

$$\underbrace{Z_L + Z_2}_{\text{in series}} \Rightarrow Z_{\text{effective}} = Z_L + Z_2$$

So, we have for the effective impedance of the above circuit,

$$\textcircled{1} \quad Z_{\text{eff}} = Z_L + (Z_0 + Z_L) Z_C / (Z_0 + Z_L + Z_C)$$

The reactance of the capacitor and the inductor are given by,

$$Z_L = i\omega L \quad (\text{reactance of the inductor})$$

$$Z_C = 1/i\omega C \quad (\text{reactance of the capacitor})$$

If the resulting impedance between the left terminals is to equal Z_0 then we have the following for eq ①,

$$Z_0 = Z_L + (Z_0 + Z_L) Z_C / (Z_0 + Z_L + Z_C)$$

$$Z_0(Z_0 + Z_L + Z_C) = Z_L(Z_0 + Z_L + Z_C) + (Z_0 + Z_L) Z_C$$

$$Z_0^2 + Z_0 Z_L + Z_0 Z_C = Z_L Z_0 + Z_L^2 + Z_L Z_C + Z_L Z_C + Z_C Z_C$$

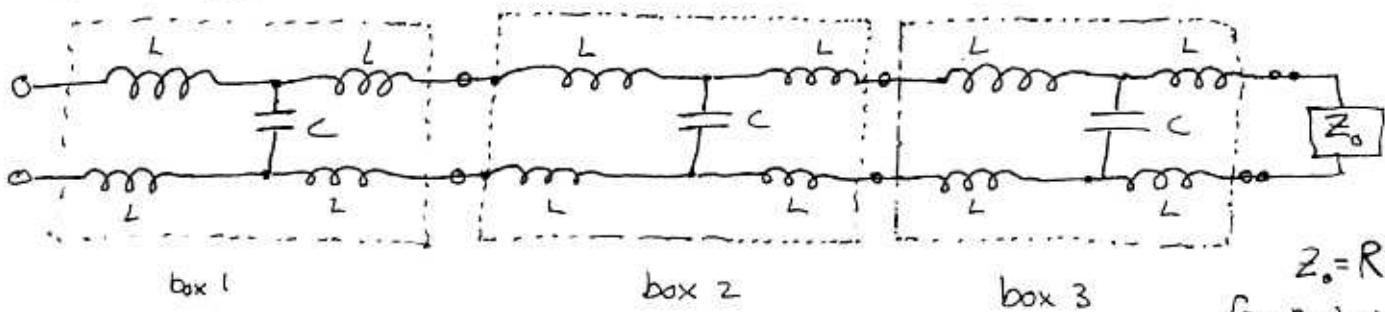
$$Z_0^2 = Z_L^2 + 2Z_L Z_C$$

$$Z_0 = \sqrt{Z_L^2 + Z_L Z_C} = \sqrt{-\omega^2 L^2 + 2L/C}$$

$$\boxed{Z_0 = \sqrt{\frac{2L}{C} - \omega^2 L^2}}$$

Z_0 is a real number when, $\frac{2L}{C} - \omega^2 L^2 > 0 \Rightarrow \omega^2 < \frac{2L}{C}$

b) Now, we wish to add a bunch of these circuits in series



If we add a chain of these boxes with the last box terminating with resistance Z_0 we will have for the last box on the right

- box 3 (Last box in chain) impedance on equals Z_0 , thus

impedance on left equals Z_0 . (from result in part a)

So, box 2 has impedance Z_0 on the right, thus the impedance on the left equals Z_0 . (using result in part a)

Lastly,

- box 1 has impedance Z_0 on the right, thus the impedance on the left equals Z_0 . (using result in part a)

So, we have the result that the input impedance equals Z_0 .

- By induction we can extend this to an infinitely long chain.

c)

$$\text{When, } \omega = \sqrt{\frac{2}{LC}}$$

$$\Rightarrow Z_0 = \sqrt{\frac{2L}{C} - \frac{2L^2}{\omega^2 C}} = 0$$

OR,

$$Z_0 = 0 \quad \text{when } \omega = \sqrt{\frac{2}{LC}}$$